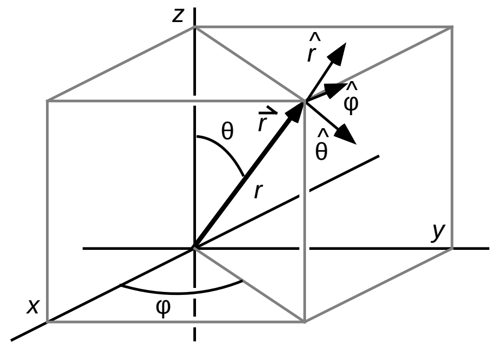
Spherical Coordinates



Transforms

The forward and reverse coordinate transformations are

$$r = \sqrt{x^2 + y^2 + z^2} \qquad x = r\sin\theta\cos\varphi$$

$$\theta = \arctan\left(\sqrt{x^2 + y^2}, z\right) \qquad y = r\sin\theta\sin\varphi$$

$$\varphi = \arctan(y, x) \qquad z = r\cos\theta$$

where we *formally* take advantage of the *two argument* arctan function to eliminate quadrant confusion.

Unit Vectors

The unit vectors in the spherical coordinate system are functions of position. It is convenient to express them in terms of the *spherical* coordinates and the unit vectors of the *rectangular* coordinate system which are *not* themselves functions of position.

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} = \hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta$$

$$\hat{\varphi} = \frac{\hat{z}\times\hat{r}}{\sin\theta} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$$

$$\hat{\theta} = \hat{\varphi}\times\hat{r} = \hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta$$

Variations of unit vectors with the coordinates

Using the expressions obtained above it is easy to derive the following handy relationships:

$$\frac{\partial\hat{r}}{\partial r} = 0$$

$$\frac{\partial\hat{r}}{\partial\theta} = \hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta = \hat{\theta}$$

$$\frac{\partial\hat{r}}{\partial\varphi} = -\hat{x}\sin\theta\sin\varphi + \hat{y}\sin\theta\cos\varphi = \left(-\hat{x}\sin\varphi + \hat{y}\cos\varphi\right)\sin\theta = \hat{\varphi}\sin\theta$$

$$\frac{\partial\hat{\varphi}}{\partial r} = 0$$

$$\frac{\partial\hat{\varphi}}{\partial\theta} = 0$$

$$\frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{x}\cos\varphi - \hat{y}\sin\varphi = -\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)$$

$$\frac{\partial\hat{\theta}}{\partial r} = 0$$

$$\frac{\partial\hat{\theta}}{\partial\theta} = -\hat{x}\sin\theta\cos\varphi - \hat{y}\sin\theta\sin\varphi - \hat{z}\cos\theta = -\hat{r}$$

$$\frac{\partial\hat{\theta}}{\partial\varphi} = -\hat{x}\cos\theta\sin\varphi + \hat{y}\cos\theta\cos\varphi = \hat{\varphi}\cos\theta$$

Path increment

We will have many uses for the path increment $d\vec{r}$ expressed in spherical coordinates:

$$d\vec{r} = d\left(r\hat{r}\right) = \hat{r}dr + rd\hat{r} = \hat{r}dr + r\left(\frac{\partial \hat{r}}{\partial r}dr + \frac{\partial \hat{r}}{\partial \theta}d\theta + \frac{\partial \hat{r}}{\partial \varphi}d\varphi\right)$$

$$= \hat{r}dr + \hat{\theta}rd\theta + \hat{\varphi}r\sin\theta d\varphi$$

Time derivatives of the unit vectors

We will also have many uses for the time derivatives of the unit vectors expressed in spherical coordinates:

$$\dot{\hat{r}} = \frac{\partial \hat{r}}{\partial r}\dot{r} + \frac{\partial \hat{r}}{\partial \theta}\dot{\theta} + \frac{\partial \hat{r}}{\partial \varphi}\dot{\varphi} = \hat{\theta}\dot{\theta} + \hat{\varphi}\dot{\varphi}\sin\theta$$

$$\dot{\hat{\theta}} = \frac{\partial \hat{\theta}}{\partial r}\dot{r} + \frac{\partial \hat{\theta}}{\partial \theta}\dot{\theta} + \frac{\partial \hat{\theta}}{\partial \varphi}\dot{\varphi} = -\hat{r}\dot{\theta} + \hat{\varphi}\dot{\varphi}\cos\theta$$

$$\dot{\hat{\varphi}} = \frac{\partial \hat{\varphi}}{\partial r}\dot{r} + \frac{\partial \hat{\varphi}}{\partial \theta}\dot{\theta} + \frac{\partial \hat{\varphi}}{\partial \varphi}\dot{\varphi} = -\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\dot{\varphi}$$

Velocity and Acceleration

The velocity and acceleration of a particle may be expressed in spherical coordinates by taking into account the associated rates of change in the unit vectors:

$$\vec{v} = \dot{\vec{r}} = \dot{\hat{r}}r + \hat{r}\dot{r}$$

$$\boxed{\vec{v} = \hat{r}\dot{r} + \hat{\theta}r\dot{\theta} + \hat{\varphi}r\dot{\varphi}\sin\theta}$$

$$\vec{a} = \dot{\vec{v}} = \dot{\hat{r}}\dot{r} + \hat{r}\ddot{r} + \dot{\hat{\theta}}r\dot{\theta} + \hat{\theta}\dot{r}\dot{\theta} + \hat{\theta}r\ddot{\theta} + \dot{\hat{\varphi}}r\dot{\varphi}\sin\theta + \hat{\varphi}\dot{r}\dot{\varphi}\sin\theta + \hat{\varphi}r\ddot{\varphi}\sin\theta + \hat{\varphi}r\dot{\varphi}\dot{\theta}\cos\theta$$

$$= \left(\hat{\theta}\dot{\theta} + \hat{\varphi}\dot{\varphi}\sin\theta\right)\dot{r} + \hat{r}\ddot{r} + \left(-\hat{r}\dot{\theta} + \hat{\varphi}\dot{\varphi}\cos\theta\right)r\dot{\theta} + \hat{\theta}\dot{r}\dot{\theta} + \hat{\theta}r\ddot{\theta}$$

$$+\left[-\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\dot{\varphi}\right]r\dot{\varphi}\sin\theta + \hat{\varphi}\dot{r}\dot{\varphi}\sin\theta + \hat{\varphi}r\ddot{\varphi}\sin\theta + \hat{\varphi}r\dot{\varphi}\dot{\theta}\cos\theta$$

$$\boxed{\vec{a} = \hat{r}\left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin\theta\right) + \hat{\theta}\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta\right) + \hat{\varphi}\left(r\ddot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + 2\dot{r}\dot{\varphi}\sin\theta\right)}$$

The del operator from the definition of the gradient

Any (static) scalar field u may be considered to be a function of the spherical coordinates r, θ, and φ. The value of u changes by an infinitesimal amount du when the point of observation is changed by $d\vec{r}$. That change may be determined from the partial derivatives as

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta + \frac{\partial u}{\partial \varphi}d\varphi .$$

But we also define the gradient in such a way as to obtain the result

$$du = \vec{\nabla} u \cdot d\vec{r}$$

Therefore,

$$\frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta + \frac{\partial u}{\partial \varphi}d\varphi = \vec{\nabla} u \cdot d\vec{r}$$

or, in spherical coordinates,

$$\frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta + \frac{\partial u}{\partial \varphi}d\varphi = \left(\vec{\nabla} u\right)_r dr + \left(\vec{\nabla} u\right)_\theta r\,d\theta + \left(\vec{\nabla} u\right)_\varphi r\sin\theta\, d\varphi$$

and we demand that this hold for any choice of dr, $d\theta$, and $d\varphi$. Thus,

$$\left(\vec{\nabla} u\right)_r = \frac{\partial u}{\partial r}, \quad \left(\vec{\nabla} u\right)_\theta = \frac{1}{r}\frac{\partial u}{\partial \theta}, \quad \left(\vec{\nabla} u\right)_\varphi = \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi},$$

from which we find

$$\boxed{\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial \varphi}}$$

Divergence

The divergence $\vec{\nabla}\cdot\vec{A}$ is carried out taking into account, once again, that the unit vectors themselves are functions of the coordinates. Thus, we have

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)\cdot\left(A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}\right)$$

where the derivatives must be taken *before* the dot product so that

$$\begin{aligned}
\vec{\nabla}\cdot\vec{A} &= \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)\cdot\vec{A} \\
&= \hat{r}\cdot\frac{\partial \vec{A}}{\partial r} + \frac{\hat{\theta}}{r}\cdot\frac{\partial \vec{A}}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\cdot\frac{\partial \vec{A}}{\partial \varphi} \\
&= \hat{r}\cdot\left(\frac{\partial A_r}{\partial r}\hat{r} + \frac{\partial A_\theta}{\partial r}\hat{\theta} + \frac{\partial A_\varphi}{\partial r}\hat{\varphi} + A_r\frac{\partial \hat{r}}{\partial r} + A_\theta\frac{\partial \hat{\theta}}{\partial r} + A_\varphi\frac{\partial \hat{\varphi}}{\partial r}\right) \\
&\quad + \frac{\hat{\theta}}{r}\cdot\left(\frac{\partial A_r}{\partial \theta}\hat{r} + \frac{\partial A_\theta}{\partial \theta}\hat{\theta} + \frac{\partial A_\varphi}{\partial \theta}\hat{\varphi} + A_r\frac{\partial \hat{r}}{\partial \theta} + A_\theta\frac{\partial \hat{\theta}}{\partial \theta} + A_\varphi\frac{\partial \hat{\varphi}}{\partial \theta}\right) \\
&\quad + \frac{\hat{\varphi}}{r\sin\theta}\cdot\left(\frac{\partial A_r}{\partial \varphi}\hat{r} + \frac{\partial A_\theta}{\partial \varphi}\hat{\theta} + \frac{\partial A_\varphi}{\partial \varphi}\hat{\varphi} + A_r\frac{\partial \hat{r}}{\partial \varphi} + A_\theta\frac{\partial \hat{\theta}}{\partial \varphi} + A_\varphi\frac{\partial \hat{\varphi}}{\partial \varphi}\right)
\end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned}
\vec{\nabla}\cdot\vec{A} &= \hat{r}\cdot\left(\frac{\partial A_r}{\partial r}\hat{r} + \frac{\partial A_\theta}{\partial r}\hat{\theta} + \frac{\partial A_\varphi}{\partial r}\hat{\varphi} + 0 + 0 + 0\right) \\
&\quad + \frac{\hat{\theta}}{r}\cdot\left(\frac{\partial A_r}{\partial \theta}\hat{r} + \frac{\partial A_\theta}{\partial \theta}\hat{\theta} + \frac{\partial A_\varphi}{\partial \theta}\hat{\varphi} + A_r\hat{\theta} + A_\theta\left(-\hat{r}\right) + 0\right) \\
&\quad + \frac{\hat{\varphi}}{r\sin\theta}\cdot\left(\frac{\partial A_r}{\partial \varphi}\hat{r} + \frac{\partial A_\theta}{\partial \varphi}\hat{\theta} + \frac{\partial A_\varphi}{\partial \varphi}\hat{\varphi} + A_r\sin\theta\,\hat{\varphi} + A_\theta\cos\theta\,\hat{\varphi} + A_\varphi\left[-\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\right]\right) \\
&= \left(\frac{\partial A_r}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{A_r}{r}\right) + \left(\frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi} + \frac{A_r}{r} + \frac{A_\theta\cos\theta}{r\sin\theta}\right) \\
&= \left(\frac{\partial A_r}{\partial r} + \frac{2A_r}{r}\right) + \left(\frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{A_\theta\cos\theta}{r\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}
\end{aligned}$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}}$$

Curl

The curl $\vec\nabla\times\vec A$ is also carried out taking into account that the unit vectors themselves are functions of the coordinates. Thus, we have

$$\vec\nabla\times\vec A=\left(\hat r\frac{\partial}{\partial r}+\frac{\hat\theta}{r}\frac{\partial}{\partial\theta}+\frac{\hat\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\times\left(A_r\hat r+A_\theta\hat\theta+A_\varphi\hat\varphi\right)$$

where the derivatives must be taken *before* the dot product so that

$$\begin{aligned}\vec\nabla\times\vec A&=\left(\hat r\frac{\partial}{\partial r}+\frac{\hat\theta}{r}\frac{\partial}{\partial\theta}+\frac{\hat\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\times\vec A\\
&=\hat r\times\frac{\partial\vec A}{\partial r}+\frac{\hat\theta}{r}\times\frac{\partial\vec A}{\partial\theta}+\frac{\hat\varphi}{r\sin\theta}\times\frac{\partial\vec A}{\partial\varphi}\\
&=\hat r\times\left(\frac{\partial A_r}{\partial r}\hat r+\frac{\partial A_\theta}{\partial r}\hat\theta+\frac{\partial A_\varphi}{\partial r}\hat\varphi+A_r\frac{\partial\hat r}{\partial r}+A_\theta\frac{\partial\hat\theta}{\partial r}+A_\varphi\frac{\partial\hat\varphi}{\partial r}\right)\\
&\quad+\frac{\hat\theta}{r}\times\left(\frac{\partial A_r}{\partial\theta}\hat r+\frac{\partial A_\theta}{\partial\theta}\hat\theta+\frac{\partial A_\varphi}{\partial\theta}\hat\varphi+A_r\frac{\partial\hat r}{\partial\theta}+A_\theta\frac{\partial\hat\theta}{\partial\theta}+A_\varphi\frac{\partial\hat\varphi}{\partial\theta}\right)\\
&\quad+\frac{\hat\varphi}{r\sin\theta}\times\left(\frac{\partial A_r}{\partial\varphi}\hat r+\frac{\partial A_\theta}{\partial\varphi}\hat\theta+\frac{\partial A_\varphi}{\partial\varphi}\hat\varphi+A_r\frac{\partial\hat r}{\partial\varphi}+A_\theta\frac{\partial\hat\theta}{\partial\varphi}+A_\varphi\frac{\partial\hat\varphi}{\partial\varphi}\right)\end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned}\vec\nabla\times\vec A&=\hat r\times\left(\frac{\partial A_r}{\partial r}\hat r+\frac{\partial A_\theta}{\partial r}\hat\theta+\frac{\partial A_\varphi}{\partial r}\hat\varphi+0+0+0\right)\\
&\quad+\frac{\hat\theta}{r}\times\left(\frac{\partial A_r}{\partial\theta}\hat r+\frac{\partial A_\theta}{\partial\theta}\hat\theta+\frac{\partial A_\varphi}{\partial\theta}\hat\varphi+A_r\hat\theta+A_\theta\left(-\hat r\right)+0\right)\\
&\quad+\frac{\hat\varphi}{r\sin\theta}\times\left(\frac{\partial A_r}{\partial\varphi}\hat r+\frac{\partial A_\theta}{\partial\varphi}\hat\theta+\frac{\partial A_\varphi}{\partial\varphi}\hat\varphi+A_r\sin\theta\hat\varphi+A_\theta\cos\theta\hat\varphi+A_\varphi\left[-\left(\hat r\sin\theta+\hat\theta\cos\theta\right)\right]\right)\\
&=\left(\frac{\partial A_\theta}{\partial r}\hat\varphi-\frac{\partial A_\varphi}{\partial r}\hat\theta\right)+\left(-\frac{1}{r}\frac{\partial A_r}{\partial\theta}\hat\varphi+\frac{1}{r}\frac{\partial A_\varphi}{\partial\theta}\hat r+\frac{A_\theta}{r}\hat\varphi\right)\\
&\quad+\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi}\hat\theta-\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}\hat r-\frac{A_\varphi}{r}\hat\theta+\frac{A_\varphi\cos\theta}{r\sin\theta}\hat r\right)\\
&=\hat r\left(\frac{1}{r}\frac{\partial A_\varphi}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}+\frac{A_\varphi\cos\theta}{r\sin\theta}\right)\\
&\quad+\hat\theta\left(-\frac{\partial A_\varphi}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi}-\frac{A_\varphi}{r}\right)\\
&\quad+\hat\varphi\left(\frac{\partial A_\theta}{\partial r}-\frac{1}{r}\frac{\partial A_r}{\partial\theta}+\frac{A_\theta}{r}\right)\end{aligned}$$

$$\boxed{\vec\nabla\times\vec A=\frac{\hat r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\varphi\sin\theta\right)-\frac{\partial A_\theta}{\partial\varphi}\right]+\frac{\hat\theta}{r\sin\theta}\left[\frac{\partial A_r}{\partial\varphi}-\sin\theta\frac{\partial}{\partial r}\left(rA_\varphi\right)\right]+\frac{\hat\varphi}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right)-\frac{\partial A_r}{\partial\theta}\right]}$$

Laplacian

The Laplacian is a scalar operator that can be determined from its definition as

$$\nabla^2 u = \vec{\nabla}\cdot\left(\vec{\nabla} u\right) = \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial}{\partial \varphi}\right)\cdot\left(\hat{r}\frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial u}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial u}{\partial \varphi}\right)$$

$$= \hat{r}\cdot\frac{\partial}{\partial r}\left(\hat{r}\frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial u}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial u}{\partial \varphi}\right)$$

$$+ \frac{\hat{\theta}}{r}\cdot\frac{\partial}{\partial \theta}\left(\hat{r}\frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial u}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial u}{\partial \varphi}\right)$$

$$+ \frac{\hat{\varphi}}{r\sin\theta}\cdot\frac{\partial}{\partial \varphi}\left(\hat{r}\frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial u}{\partial \theta} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial u}{\partial \varphi}\right)$$

With the help of the partial derivatives previously obtained, we find

$$\nabla^2 u = \hat{r}\cdot\left(\hat{r}\frac{\partial^2 u}{\partial r^2} - \frac{\hat{\theta}}{r^2}\frac{\partial u}{\partial \theta} + \frac{\hat{\theta}}{r}\frac{\partial^2 u}{\partial \theta\partial r} - \frac{\hat{\varphi}}{r^2\sin\theta}\frac{\partial u}{\partial \varphi} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial^2 u}{\partial \varphi\partial r}\right)$$

$$+ \frac{\hat{\theta}}{r}\cdot\left(\hat{\theta}\frac{\partial u}{\partial r} + \hat{r}\frac{\partial^2 u}{\partial r\partial \theta} - \frac{\hat{r}}{r}\frac{\partial u}{\partial \theta} + \frac{\hat{\theta}}{r}\frac{\partial^2 u}{\partial \theta^2} - \frac{\hat{\varphi}\cos\theta}{r\sin^2\theta}\frac{\partial u}{\partial \varphi} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial^2 u}{\partial \varphi\partial \theta}\right)$$

$$+ \frac{\hat{\varphi}}{r\sin\theta}\cdot\left(\hat{\varphi}\sin\theta\frac{\partial u}{\partial r} + \hat{r}\frac{\partial^2 u}{\partial r\partial \varphi} + \frac{\hat{\varphi}\cos\theta}{r}\frac{\partial u}{\partial \theta} + \frac{\hat{\theta}}{r}\frac{\partial^2 u}{\partial \theta\partial \varphi} - \frac{\hat{r}\sin\theta + \hat{\theta}\cos\theta}{r\sin\theta}\frac{\partial u}{\partial \varphi} + \frac{\hat{\varphi}}{r\sin\theta}\frac{\partial^2 u}{\partial \varphi^2}\right)$$

$$= \left(\frac{\partial^2 u}{\partial r^2}\right) + \left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) + \left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}\right)$$

$$= \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) + \left(\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial \theta}\right) + \left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$$

Thus, the Laplacian operator can be written as

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}}$$