and spectral and finite element techniques) and schemes which are formally conservative of lower-order properties (mass, energy, enstrophy, etc.), even these approaches often fail to meet some of the above criteria (e.g., positivity of fields). Some of these points are reviewed in detail in Arakawa and Lamb (1977), O'Brien (1986) and Rood (1987).

11.4 Simple models of the ocean circulation

11.4.1 The wind-driven ocean

Although climate system models will undoubtedly require sophisticated ocean model components, much physical intuition about the ocean circulation may be gained from simpler models. Consider the classical example of a homogeneous ocean ($\rho = \rho_o =$ constant) of uniform depth H driven at its surface by a specified wind stress $\boldsymbol{\tau}_s = (\tau_{s_x}, \tau_{s_y})$, as shown in Fig. 11.1. In this idealized setting, and after appropriate simplification of the equations of motion, the steady response of the ocean to the applied stress can be determined directly. Among other things, this reveals the important role of surface and lateral boundary layers.

On an f-plane (i.e., with $f = f_o$), and, for the moment, ignoring temporal and lateral variations ($\frac{\partial}{\partial t} = \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$), an important integral property of the surface *Ekman boundary layer* may be determined as described in Chapter 4, Sec. 4.4. In particular, the continuity equation, together with the requirement that vertical velocity vanish at the surface (presumed to be a rigid lid), implies that w vanishes everywhere; hence, as given in (4.2), the momentum equations become:

$$-f_o v = \frac{1}{\rho_o}\frac{\partial \tau_{s_x}}{\partial z} \tag{11.25a}$$

and

$$f_o u = \frac{1}{\rho_o}\frac{\partial \tau_{s_y}}{\partial z} \,. \tag{11.25b}$$

Among the properties of the Ekman layer is that the integrated mass flux in the surface Ekman layer is proportional to the applied stress:

$$\int_{-\infty}^{0} (\rho_o v) dz = -\tau_{s_x}/f_o \tag{11.26a}$$

$$\int_{-\infty}^{0} (\rho_o u) dz = \tau_{s_y}/f_o \,. \tag{11.26b}$$

and that there is a net vertical velocity, or *Ekman pumping*, at the base of the layer which is related to the curl of the wind stress:

$$w_E = \mathbf{k} \cdot \nabla \times \frac{\boldsymbol{\tau}_s}{\rho_o f_o} \,, \tag{11.27}$$

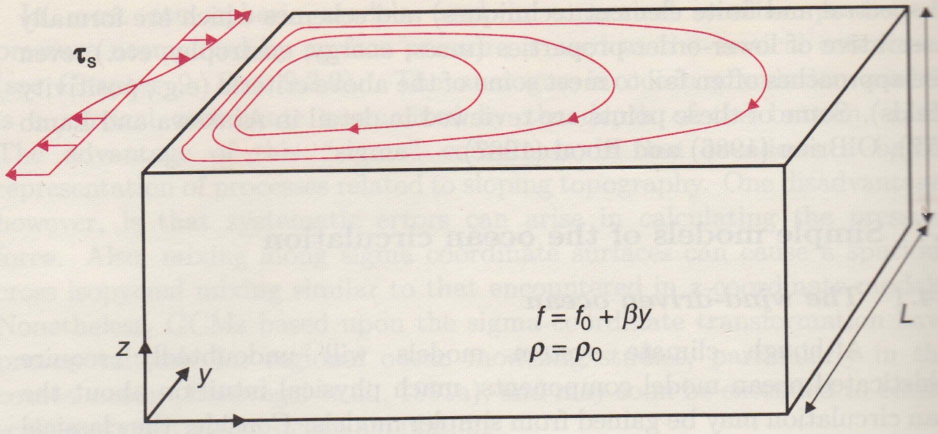

Fig. 11.1 Schematic of the barotropic model of the wind-driven circulation of the subtropical gyre.

see also Eq. (4.5). Expressions analogous to (11.27) may also be derived for the net effects of an *Ekman bottom boundary layer*. There, the Ekman pumping is proportional to the vorticity of the interior flow. See, for example, the complete treatment given by Pedlosky (1987a). The vertical mass fluxes at the ocean surface and bottom, though weak, are an essential element of the ocean's response to applied wind stresses.

In Cartesian geometry, and with a constant eddy viscosity closure, the HPE for a homogeneous ocean can be scaled to yield a simple, single equation which contains much information about the response of the ocean to an imposed wind stress (Pedlosky, 1987a). For a basin-scale flow in the interior of the ocean, and with L, H and U_o representing typical length, height, and velocity scales respectively, the resulting scaled equations are characterized by several nondimensional parameters: the Rossby number ($Ro = U_o/f_oL$), nondimensional beta ($\hat{\beta} = \beta L/f_o$), and the horizontal and vertical Ekman numbers ($E_h = \nu_h/f_oL^2$, $E_v = \nu_v/f_oH^2$), where, $f_o = 2\Omega \sin\phi_o$ and $\beta = f_o \cot\phi_o/a$. If it is now assumed $Ro \approx E_v^{\frac{1}{2}} \approx \hat{\beta} < 1$ and $E_h << 1$, the lowest-order interior flow is found to be geostrophic, horizontally nondivergent, and depth independent.

At the next order in Rossby number, the evolutionary equation is:

$$\frac{d}{dt}\left[\zeta + \left(\frac{\hat{\beta}}{Ro}\right) y\right] = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\left[\zeta + \left(\frac{\hat{\beta}}{Ro}\right) y\right] \qquad (11.28)$$

$$= w_{z=0} - w_{z=-H} + \left(\frac{E_h}{Ro}\right)\nabla^2\zeta \;,$$

where the relative vorticity $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. Equation (11.28), a form of the well-known *barotropic vorticity equation* (BVE), shows how the evolution of the vorticity in the interior of the ocean is directly influenced by vorticity produced by vertical velocities at the surface ($w_{z=0}$) and/or bottom ($w_{z=-H}$). Using the values for w_E at the top and bottom, the BVE, in dimensional form, is then:

$$\frac{d}{dt}(\zeta + \beta y) = \mathbf{k} \cdot \nabla \times \left(\frac{\tau_s}{\rho_o H} \right) - r\zeta + \nu_h \nabla^2 \zeta ,$$

where r^{-1} is a frictional time scale related to the vertical eddy viscosity (ν_v). Since the lowest-order flow field is horizontally nondivergent, a streamfunction $\psi(x, y)$ such that $(u, v) = (-\partial\psi/\partial y,\ \partial\psi/\partial x)$ may also be introduced. Then:

$$\left[\frac{\partial}{\partial t} + J(\psi, \ ) \right] (\nabla^2 \psi + \beta y) = \mathbf{k} \cdot \nabla \times \frac{\tau_s}{\rho_o H} - r\nabla^2 \psi + \nu_h \nabla^4 \psi \qquad (11.29)$$

where J is the Jacobian.

In the ocean interior, the dominant terms in (11.29) reflect a balance between the input of vorticity by the wind stress curl and the vorticity tendency associated with north–south movement on the β-plane:

$$\beta v = \mathbf{k} \cdot \nabla \times \frac{\tau_s}{\rho_o H} . \qquad (11.30)$$

This is called the *Sverdrup balance* (see Chapter 4, Sec. 4.5). Suppose the wind stress takes the form $\tau_{s_x} = -\tau_o \cos(\pi y/L)$ in a square basin of length L. Then, in the interior, the north–south velocity

$$\frac{\partial \psi}{\partial x} = v = -\left(\frac{\tau_o \pi}{\rho_o H L \beta} \right) \sin\left(\frac{\pi y}{L} \right) . \qquad (11.31)$$

However, the circulation pattern is not fully determined. [To see this, try to integrate (11.31) to determine a $\psi(x, y)$ field that satisfies the constraint that ψ vanish simultaneously at $x = 0$ and L, corresponding to the requirement that there be no normal flow through any point on either the eastern or western boundaries.] In particular, since the interior flow is directed southwards at any interior latitude, a boundary layer must exist on one of the meridional walls to close the mass flux. By considering conservation of vorticity, it can be shown that the boundary layer must exist on the western wall (Stommel, 1965). With bottom friction of the Ekman sort, the dominant balance in the western boundary layer is

$$\beta v = \beta \psi_x \approx r \psi_{xx} ,$$

with the associated boundary layer structure

$$\psi \approx \exp(-\beta x / r) .$$

If the boundary layer thickness $(r/\beta) << L$, the total solution can then be written:

$$\psi(x, y) = \frac{\pi \tau_o}{H \beta \rho_o} \left[\frac{(L - x)}{L} - e^{-\beta x / r} \right] \sin(\pi y / L) , \qquad (11.32)$$

in which the Sverdrup (interior) and boundary layer circulations are combined. This is the *Stommel model* of the wind-driven circulation (Stommel, 1948). Analogous analytic models incorporating the effects of lateral friction and weak nonlinearity have been examined by Munk (1950) and Charney (1955a). The earliest numerical extensions of this work were due to Bryan (1963).

11.4.2 Box models of the thermohaline circulation

The simple models for the wind-driven circulation described in the previous section assume that buoyancy variations play little or only a passive role in the dynamics. Despite this drastic assumption, they are quite successful in representing the basic patterns of the circulation and provide the underpinning of much of the theory of the ocean general circulation. For climate applications, however, it is the spatial and temporal variations of the temperature and salinity distributions of the ocean and its capacity for heat storage and transport that are of primary concern. The determination of the thermohaline (joint effects of heat and salt on buoyancy) driven circulation and the reciprocal effects of the circulation on the distribution of water mass properties are difficult problems for several reasons. First and foremost is the essential nonlinearity of the system. The models of the previous section could be obtained through a systematic scale analysis and linearization of the governing equations. In considering the thermohaline circulation, the advection of heat and salt by the circulation is central to the problem and cannot be neglected. A further complication arises from the difference between the form of the surface forcing for temperature and salinity, as discussed in Sec. 11.2.3. As a result of the different mathematical structure of the boundary conditions on heat and salt (often referred to in the literature as mixed boundary conditions) the problem cannot be reduced to one in a single buoyancy variable. Additional complications in modeling the thermohaline circulation arise from the nonlinear equation of state for sea water and the presence of double-diffusive phenomena.

An alternative to the formal mathematical derivation of simplified models from the full equations of motion is to pose a conceptual model or simple physical analog for the system or processes being considered. This approach has been fruitfully exploited in developing our understanding of the dynamics of the thermohaline circulation. The consequences of the difference in the nature of the feedbacks between surface temperature and salinity and their respective surface forcings were first explored by Stommel (1961) using a very simple model consisting of two well-mixed reservoirs connected by pipes. In particular, he showed that the thermohaline circulation may have multiple equilibria for a given surface forcing distribution.

These ideas are illustrated using a slightly modified version of the model (Fig. 11.2), as described by Marotzke (1989). The two reservoirs are taken to represent equatorial and polar regions of the ocean. In light of the discussion