ADJOINT PERTURBATION METHOD APPLIED TO TWO-STREAM RADIATIVE TRANSFER

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Abstract—This paper describes a computationally efficient method for solving the plane parallel equation of radiative transfer for the two-stream fluxes based on the adjoint perturbation formulation. Analytical results for the perturbed fluxes are presented for a single layer atmosphere containing both solar and thermal sources. Simple linear and exponential corrections to the base state fluxes are explored. For the solar radiative transfer problem, the exponential form of the perturbation correction can accommodate deviations exceeding 400% in the base state optical properties while maintaining accuracy to within a few percent. For thermal radiative transfer, the linear form of perturbation relation is the more accurate, but unlike the solar problem, deviations from the base state optical properties must remain relatively small (less than 20%) if the errors in the computed fluxes are to remain within a few percent of the true fluxes. The method is applied to the calculation of broadband solar fluxes in a layer of scatterers embedded in an absorbing gas, where the absorption is modeled via the k-distribution method. © 1998 Published by Elsevier Science Ltd. All rights reserved

1. INTRODUCTION

Over the past few years, the demand for simple, robust and computationally fast methods for solving various radiative transfer problems has increased. This increase in demand is largely motivated by the topic of climate and global change. In this context, problems of interest range from radiative transfer through clouds characterized by highly anisotropic scattering, radiative transfer through clear atmospheres involving mixes of Rayleigh scattering and molecular absorption, the effects of increased absorption associated with increases in selected greenhouse gases, scattering by optically thin atmospheres comprised of aerosols and cloud particles as well as the complex effects of cloud geometry on radiative transfer.

The time required for radiative transfer computations is excessive for even the simplest of radiative transfer schemes when incorporated into Global Circulation Models (GCMs) that attempt to simulate the complex climate system. The problem stems from the nature of the calculation of broadband fluxes which involves repetitive summations of many radiative transfer solutions with input parameters that often have similar values. The desire is to develop highly efficient techniques to carry out these broadband calculations. By reducing significantly the computation time devoted to modeling of radiative transport while maintaining numerical accuracy, the potential of improving other critical processes can be realized.

An approach that has historically been used to streamline this process of calculation is based on the use of the perturbation form of the radiative transfer equation. The Curtis matrix method is an example of a perturbation approach that is based on some form of simplification of the radiative transfer equation. There are, however, many other methods that are more ad hoc in nature. With the complexity of climate models and the demands for more physical and accurate radiative transfer models, perturbation methods have again emerged as a viable way of parameterizing radiative transfer.

In this paper we extend the perturbation method introduced to the atmospheric science community by Gerstl, Box et al and others to the problem of broadband radiative transfer. The approach presented is based on a theoretical method successfully used in neutron transport pro-
The foundation of the perturbation approximation used here has been described by Box et al. where the flux $F$ at level $z$ is approximated as

$$F(z) = \langle \mathcal{F}_o, Q_f \rangle - \langle \mathcal{F}_o, \Delta^2 G_o \Delta^2 F_o \rangle - \langle \mathcal{F}_o, \Delta^2 G_o \Delta^2 G_o \Delta^2 F_o \rangle + \ldots$$

(1)

where $F_o$ is the flux defined for a given base state (indicated hereafter by the subscript "o"). $F$ is the required flux for an atmosphere that differs from the base state, $\mathcal{F}_o$ is the solution of the adjoint of the equation of transfer for the base state, $\Delta^2$ is the perturbation of the radiative transfer operator (defined below) and $G_o$ is the Green's function of the base state. The function $Q_f$ is the source function of the perturbed equation of transfer. To a first order, we will retain only the first of these expansion terms so that Equation (1) becomes:

$$F(z) \approx \langle \mathcal{F}_o, Q_f \rangle - \Delta F$$

(2)

where

$$\Delta F = \langle \mathcal{F}_o, \Delta^2 F_o \rangle$$

is the flux difference between the two states. The braces indicate the inner product which is defined as

$$\langle f, g \rangle = \int_x^z (f^+ g^+ + f^- g^-) \, dz'$$

(3)

where

$$f = \begin{bmatrix} f^+ \\ f^- \end{bmatrix}, \quad g = \begin{bmatrix} g^+ \\ g^- \end{bmatrix}$$

(e.g., Bronson and Bellman). The symbols $x$ and $z$ designate the positions of the base and upper boundary of the medium respectively.

As mentioned above, the general idea for this approach was suggested for atmospheric transfer problems by Gerstl and explored by Box and colleagues for selected "n-stream" radiance problems. The principal purpose of this paper is to present the analytical form of the perturbation two-stream flux, explore the relation between this perturbation flux and the desired fluxes $F$, and assess the performance of the approach for two specific problems.

This paper begins with an outline of two-stream radiative transfer and adjoint transfer theory. Solutions are given for the problem of a scattering layer illuminated from above by a collimated source and for the problem of a layer illuminated from below by a diffuse source. This sets the stage for application of the perturbation approach to problems of both solar and infrared radiative transfer. These analytical solutions for the perturbation equations are given in Section 3. Comparisons of these solutions with equivalent two-stream solutions are given in Section 4. The perturbation approach is further illustrated in Section 5 where it is applied to the problem of broadband solar radiation transfer through a continuum scattering layer embedded within an absorbing gaseous atmosphere. The perturbations are introduced as a rapid way of calculating broadband fluxes via the correlated k-distribution method. The results and conclusions of the paper are summarized in the final section of the paper.

2. THE ADJOINT OF TWO-STREAM RADIATIVE TRANSFER

The geometry considered in this paper and parameters relevant to this geometry are given in Fig. 1. The medium considered is a single, vertically homogeneous layer which scatters, absorbs and emits radiation. The layer extends from the lower level $x$ to an upper level $z$ with an intermediate reference level $y$. An arbitrary level between $y$ and either boundary is represented by $w$. Extension of the approach described in this paper to multi-layered media is the subject of a forthcoming study.

The two-stream equations relevant to this medium are

$$\mathcal{L}_o F_o = Q_o$$

and

$$\mathcal{L} F = Q_f$$

(4a)
where \( F_0 \) is a flux vector

\[
F_0 = \begin{pmatrix} F^+_0 \\ F^-_0 \end{pmatrix}
\]

of upwelling \((F^+_0)\) and downwelling \((F^-_0)\) fluxes at level \( y \). The source vector corresponding to the base state is

\[
Q_0 = \begin{pmatrix} Q^+_0 \\ Q^-_0 \end{pmatrix}
\]

and the two-stream base state operator is

\[
\mathcal{L}_0 = \frac{d}{dy} \begin{pmatrix} -t & r \\ -r & t \end{pmatrix}.
\] (4b)

In Equation (4a) we seek a solution to \( \mathcal{L}F = Q_0 \) in terms of the base state solution to the equation of transfer, i.e., \( F_0 \). That connection is established in (1). The 2 \times 2 matrix of coefficients appropriately defines the attenuation matrix

\[
A = \begin{pmatrix} -t & r \\ -r & t \end{pmatrix}.
\] (5a)

The specific form of the equation coefficients in (4b), namely \( t \) and \( r \), and the form of the source functions considered in this paper are those of the Eddington approximation, i.e.,

\[
t = \sigma_{\text{ext}} \frac{7}{4} - \frac{\sigma_{\text{sca}}}{4} (4 + 3g)
\]

\[
r = -\sigma_{\text{ext}} \frac{1}{4} + \frac{\sigma_{\text{sca}} (4 - 3g)}{4}
\] (5b)

\[
Q^+_0 = \frac{\sigma_{\text{sca}}}{4} [2 - 3g\mu_x]
\]

\[
Q^-_0 = \frac{\sigma_{\text{sca}}}{4} [2 + 3g\mu_x]
\] (5c)

where \( g \) is the asymmetry parameter, \( \theta_0 \), the solar zenith angle \((\mu_0 = \cos(\theta_0))\), \( \sigma_0 = \sigma_{\text{sca}}/\sigma_{\text{ext}} \) the single scatter albedo and \( \sigma_{\text{ext}} = \sigma_{\text{sca}} + \sigma_{\text{abs}} \) is the volume extinction coefficient with \( \sigma_{\text{sca}} \) and \( \sigma_{\text{abs}} \) the volume scattering and absorption coefficients respectively. The two-stream solutions are
therefore defined by 5 parameters, namely $\sigma_{h,\infty}$, $\sigma_{abs}$, $g$, $\theta_0$, and $z-x$, the geometric thickness of the layer. For completeness, the general solution to (4a) and (4b) is outlined in Appendix A, although the solution can be found elsewhere (e.g., Meador and Weaver).

2.1. Adjoint two-stream equations

The adjoint form of the two-stream equations can be shown to be:

$$\mathcal{L}_o^\dagger \mathcal{F}_o(y,w) = R(y,w)$$

where

$$\mathcal{L}_o^\dagger = -\frac{d}{dy} - A^T,$$

$\mathcal{F}_o(y,w)$ is the vector of adjoint fluxes

$$\mathcal{F}_o(y,w) = \left( \begin{array}{c} \mathcal{F}_o^+(y,w) \\ \mathcal{F}_o^-(y,w) \end{array} \right),$$

$A^T$ is the transpose of $A$ whose elements are given by (5b) and $R(y,w)$ is the response function, or adjoint source function. The response function is defined so that its inner product with the flux produces some desired effect or observable,

$$E = \langle R, F_o \rangle$$

In this paper, $E$ will be taken to mean the emergent hemispheric fluxes $F^+(z)$ and $F^-(x)$ at the boundaries $z$ and $x$, i.e., for $y = x$ or $y = z$. Equation (6) has been derived by several authors (e.g., Box et al). The result is obtained by using integration by parts to evaluate the inner product $\langle \mathcal{F}_o, \mathcal{L}_o F_o \rangle$. The expression outside of the integral, known as the bilinear concomitant, is set to zero, from which the adjoint boundary conditions derive. Unlike the physical flux, the adjoint flux is not a level quantity but rather a layer quantity. It is a distribution function which yields the physical fluxes when integrated with the source function over the relevant layer of interest. In this way its usage is similar to the standard Green's function but differs by being the solution to the adjoint equation subject to adjoint boundary conditions.

The response functions for upwelling and downwelling fluxes at these boundaries are

$$R^+(z,w) = \begin{pmatrix} \delta(z-w) \\ 0 \end{pmatrix}, \quad R^-(x,w) = \begin{pmatrix} 0 \\ \delta(x-w) \end{pmatrix}$$

respectively where $\delta(y-w)$ is the Dirac-delta function. These response functions may be written in the more compact form as

$$R(y,w) = \begin{pmatrix} \delta(y-w)h_+ \\ \delta(y-w)h_- \end{pmatrix}$$

for the flux at an arbitrary level $y$ where $h_{\pm}$ is the indicator function which is zero or one depending upon whether the upwelling (+) or downwelling (-) fluxes are desired.

The interpretation of (9) in (6a) is that two equations must be solved for each pair of adjoint fluxes, one with $R^+$ as a source and the other with $R^-$. Such solutions yield the adjoint two-stream fluxes $\mathcal{F}_o(y,w)$ from which the desired flux follows from the equalities:

$$F_o = \langle \mathcal{F}_o, \mathcal{L}_o F_o \rangle = \langle \mathcal{F}_o, Q_o \rangle = \langle \mathcal{L}_o^\dagger \mathcal{F}_o, F_o \rangle,$$

since in the last equality, the adjoint equation is set equal to the Dirac-delta (response) function. We find that the flux is mathematically identical to the general solution of the two-stream equations given in Appendix A.

2.2. General solutions

In developing the solution of the adjoint two-stream equation, we follow closely the approach described in Appendix A. For this purpose, it is convenient to introduce the matrix exponential
where the elements of the matrix are again given in Appendix A Multiplying (6a) by this matrix yields

$$\frac{d e^{A\gamma} \mathcal{F}_0(y,w)}{dy} = -e^{A\gamma} R(y,w)$$

which can be integrated to obtain the general solution

$$\begin{bmatrix} \mathcal{F}^+_0(y,w) \\ \mathcal{F}^-_0(y,w) \end{bmatrix} = \begin{bmatrix} m_{++}(y) & m_{+-}(y) \\ m_{+\pm}(y) & m_{--}(y) \end{bmatrix} \begin{bmatrix} b^+ \\ b^- \end{bmatrix} \begin{bmatrix} H(y-w)h_+ \\ H(y-w)h_- \end{bmatrix}$$

where $b^+$ are related to the boundary conditions as shown in the two examples below and where $H(x)$ is the Heaviside function. We now illustrate this solution as applied to two particular problems identified earlier. In the first of these problems, vacuum boundary conditions (i.e., zero incident fluxes and zero emergent adjoint fluxes) are applied and the source of radiation within the layer is exponential as defined in the Appendix A. The second problem is that of a single layer illuminated by an isotropic source from below with an internal source as defined by (A12). The immediate relevance of these problems to radiative transfer in the Earth's atmosphere was noted previously.

2.3. Example 1: solutions for direct illumination

The adjoint of this problem is schematically portrayed in Fig. 2. The vacuum boundary conditions corresponds to the case of zero emergent adjoint flux from the boundaries of the layer, namely

$$\mathcal{F}^+_0(x,x) = 0$$
$$\mathcal{F}^-_0(x,x) = 0$$

and we assume hereafter for convenience that the lower boundary occurs at $x = 0$. Given these conditions, (12) can be rearranged to produce general solutions

$$\mathcal{F}^-_0(0,0) = h^+ + \mathcal{A}(z,0)h^-$$

Fig. 2. This figure illustrates the difference between the forward problem (left) and the adjoint (right). Light arrows represent inputs, or boundary conditions. Dark arrows are outputs. Outputs in the forward problem are physical fluxes. Adjoint fluxes are not physical, but are layer quantities that are used to determine the physical fluxes.
\[ \mathcal{F}_o(z,z) = -h^- - \mathcal{R}(0,z)h^+ \] (13)

where \( \mathcal{R}(0,z) \) and \( \mathcal{R}(z,0) \) are the diffuse reflectance functions for the layer extending from \( 0 \) to \( z \) as defined in Appendix A. Because of the vertical uniformity of the medium, the conditions of reciprocity require that \( \mathcal{R}(0,z) = \mathcal{R}(z,0) \).

2.4. Example 2: solutions for conditions of diffuse illumination

For general conditions of diffuse illumination at the boundaries, the following bilinear concomitant must be satisfied (Box et al.):

\[ \mathcal{F}_o^+(z,z) \mathcal{F}_o^+(x,x) - \mathcal{F}_o^-(x,x) \mathcal{F}_o^-(z,z) - \mathcal{F}_o^-(x,x) \mathcal{F}_o^+(z,z) = 0. \]

Diffuse illumination at the lower boundary requires that \( \mathcal{F}_o^+(z,z) = 0, \mathcal{F}_o^-(z,z) = 0 \) and \( \mathcal{F}_o^+(x,x) = F_g \) where \( F_g \) is the flux specified at \( y = x \). We obtain

\[ \mathcal{F}_o^-(x,x) = -\mathcal{F}_o^+(x,x)\left( \frac{F_g}{\mathcal{F}_o^-(x,x)} \right). \] (14)

The general solution for the adjoint flux at level \( y \) follows directly from integration of (6),

\[ \left( \begin{array}{c} \mathcal{F}_o^+(y,w) \\ \mathcal{F}_o^-(y,w) \end{array} \right) = \left( \begin{array}{cc} m_+(y) & m_-(y) \\ m_-(y) & m_+(y) \end{array} \right) \left( \begin{array}{c} C_+^+ \\ C_+^- \end{array} \right) - \left( \begin{array}{cc} m_+(y-w)h^+ + m_-(y-w)h^- \\ m_-(y-w)h^+ + m_-(y-w)h^- \end{array} \right) \] (15a)

where

\[ C_+^+[x,z,w,h^+,h^-] = \frac{F_o^+(x)(m_+(z-w)h^+ + m_-(z-w)h^-)H(z-w)}{F_o^-(x)m_+(z) - F_g m_-^-(z)} \]
\[ C_+^-[x,z,w,h^+,h^-] = \frac{-F_o^+(x)(m_+(z-w)h^+ + m_-(z-w)h^-)H(z-w)}{F_o^-(x)m_+(z) - F_g m_-^-(z)} \] (15b)

When evaluated at the boundary, the adjoint flux again can be expressed in terms of global reflection and transmission functions according to

\[ \mathcal{F}_o^+(0,0) = \frac{F_o^-(0)(h^+ + \mathcal{R}(0,z)h^-)}{F_o^-(0) - F_g \mathcal{R}(0,z)} \]
\[ \mathcal{F}_o^-(z,z) = \frac{(-\mathcal{R}(z,0)F_o^-(0) - (\mathcal{F}_o^+(z,0) - \mathcal{R}(z,0))F_g)h^+ - h^-}{F_o^-(0) - F_g \mathcal{R}(z,0)} \] (15c)

3. THE TWO-STREAM PERTURBATION EQUATIONS

First order perturbations are given by (2) and are defined by the perturbation operator \( \Delta \mathcal{L} \) which is formed by taking the difference between the perturbation attenuation matrix and the matrix defined by the (unprimed) base state parameters. This operator may be written as:

\[ \Delta \mathcal{L} = \left( \begin{array}{cc} \Delta \mathcal{L}_r & \Delta \mathcal{L}_s \\ -\Delta \mathcal{L}_s & \Delta \mathcal{L}_r \end{array} \right) \] (16)

For convenience we consider perturbations only in \( \sigma_{scs} \) and \( \sigma_{abs} \) and introduce these in the form

\[ \sigma_{abs}' = (1 + \delta_{abs})\sigma_{abs}, \]
\[ \sigma_{scs}' = (1 + \delta_{scs})\sigma_{scs} \] (17)

where \( \delta_{abs} \) and \( \delta_{scs} \) are perturbation factors which may be varied arbitrarily. Perturbations can either be negative or positive. Negative perturbations signify a reduction in the base state scat-
Table 1. The functions and parameter definitions of the adjoint perturbation solution for a single plane-parallel layer. In
applications, the functions and parameters would be calculated only once

<table>
<thead>
<tr>
<th>Function</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1(k,\omega,z)$</td>
<td>$\frac{e^{+\text{i}kz}+e^{-\text{i}kz}}{2\text{k}}$</td>
</tr>
<tr>
<td>$\phi_2(k,\omega,z)$</td>
<td>$(z-w)e^{\text{i}kz}$</td>
</tr>
<tr>
<td>$\phi_3(k,\omega,z)$</td>
<td>$\frac{e^{+\text{i}kz}+e^{-\text{i}kz}}{m_+^2(z-\text{i})}$</td>
</tr>
<tr>
<td>$\phi_4(k,\omega,z)$</td>
<td>$\frac{e^{+\text{i}kz}+e^{-\text{i}kz}}{m_-^2(z-\text{i})}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>$(k-\text{i})/2k$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$(k+\text{i})/2k$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\frac{\text{e}}{\text{i}}$</td>
</tr>
</tbody>
</table>

Perturbation or absorption. For example, $\delta_{\text{abs}} = -0.5$ is a 50% reduction in the absorption cross section, while $\delta_{\text{abs}} = -1$ denotes the absence of an absorption cross section. Positive perturbations increase the absorption. Similar considerations apply to the scattering cross section. When $\delta_{\text{abs}}$ or $\delta_{\text{scat}}$ are zero, no perturbations are present. Hence the perturbed fluxes are identical to those of the base state.

If only $\sigma_{\text{abs}}$ is perturbed, then the elements of the perturbation matrix are given by:

$$
\Delta t = \frac{7}{4} \delta_{\text{abs}} \sigma_{\text{abs}}
$$

$$
\Delta r = -\frac{1}{4} \delta_{\text{abs}} \sigma_{\text{abs}}, (18)
$$

thus defining the perturbation operator $\Delta \mathcal{L}$ by virtue of (16).

With analytical expressions for $\Delta \mathcal{L}$ such as (18) and with analytic expressions for both the adjoint and physical fluxes, analytical expressions for the perturbation fluxes can be readily obtained by evaluating the inner product. Although straightforward, this proves cumbersome. Results for the two example problems are given with derivation details omitted for brevity.

3.1. Perturbations for direct illumination

The perturbed fluxes $\Delta F(x)$ may be written compactly as a sum of products of factors:

$$
\Delta F^{\text{+}}(z) = \frac{1}{m_+^2(z)} \sum_{k=1}^{6} c_k \psi_k(x) - \sum_{k=1}^{5} c_k q_k(z)
$$

(19a)

$$
\Delta F^{-}(x) = \frac{m_-^2(z)}{m_+^2(z)} \sum_{k=1}^{6} c_k \psi_k(x) - \sum_{k=1}^{5} c_k q_k(z)
$$

(19b)

Functions $\psi_k(x), q_k(z)$ and coefficients $c_k$ for this problem are given in Tables 1 and 2.

Table 2. The coefficients required to perform adjoint perturbations calculations for the solar source functions. With the exception of coefficients $c(1)$ to $c(6)$, all of the other coefficients are calculated once

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>$\delta t A_3-\delta t A_1$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$\delta r A_4-\delta r A_2$</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$(\delta b_1-\delta b_2) \times \delta r + \delta b$</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$(\delta b_1-\delta b_2) \times \delta t + \delta b$</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$\delta t A_4-\delta r A_2$</td>
</tr>
<tr>
<td>$c_6$</td>
<td>$(\delta b_2-\delta b_1) \times \delta t + \delta t$</td>
</tr>
</tbody>
</table>

Solar source: $A_1 = -(p_1 - \frac{m_+^2(z)}{m_{+1}(z)}) (\beta_{11} X_c^+ - \beta_{12} X_c^-) \text{e}^{-\text{i}z} - \frac{m_{+1}(z)}{m_{+1}(z)} \beta_{11} (t-\text{i}) (k^2-\delta^2)$

$A_2 = -(p_2 - \frac{m_+^2(z)}{m_{+1}(z)}) (\beta_{11} X_c^+ - \beta_{12} X_c^-) \text{e}^{-\text{i}z} + \frac{m_{+1}(z)}{m_{+1}(z)} \beta_{12} (t-\text{i}) (k^2-\delta^2)$

$A_3 = (p_3 + \frac{m_+^2(z)}{m_{+1}(z)}) (\beta_{11} X_c^+ - \beta_{12} X_c^-) \text{e}^{-\text{i}z} - \frac{m_{+1}(z)}{m_{+1}(z)} \beta_{12} (t-\text{i}) (k^2-\delta^2)$

$A_4 = (p_4 + \frac{m_+^2(z)}{m_{+1}(z)}) (\beta_{11} X_c^+ - \beta_{12} X_c^-) \text{e}^{-\text{i}z} - \frac{m_{+1}(z)}{m_{+1}(z)} \beta_{12} (t-\text{i}) (k^2-\delta^2)$
We chose an exponential approximation that relates the corrected flux to the base state flux and perturbation flux,

\[ F^+(z) \approx \langle \mathcal{F}_o, Q_f \rangle^+ \exp \left[ -\frac{\Delta F^+(z)}{\langle \mathcal{F}_o, Q_f \rangle^+} \right] \]  

(21a)

\[ F^-(x) \approx \langle \mathcal{F}_o, Q_f \rangle^- \exp \left[ -\frac{\Delta F^-(x)}{\langle \mathcal{F}_o, Q_f \rangle^-} \right] \]  

(21b)

since Box et al show that this is an improvement over the linear relationship obtained by retaining the first two terms of (1). The quantities \( \langle \mathcal{F}_o, Q_f \rangle^+ \) and \( \langle \mathcal{F}_o, Q_f \rangle^- \) are the inner products of the perturbed source function and the upwelling and downwelling adjoint fluxes respectively. To accelerate the computations, equations (21a and 21b) were modified by substituting \( F^+(z) \) for \( \langle \mathcal{F}_o, Q_f \rangle^+ \). This is equivalent to stating that the differences between \( Q_o \) and \( Q_f \) are small, or alternatively, that the parameters constituting \( \mathcal{L}_o \) contribute mostly to the accuracy in the fluxes. We acknowledge that greater accuracy is attainable, but since \( \langle \mathcal{F}_o, Q_f \rangle^- \) increases the expense of the computation, we explored the utility of the approximation indicated. Further improvement on this approximation is the topic of future work.

3.2. Perturbations for diffuse illumination

For media with internal sources specified by the Planck function and illuminated at the lower boundary by isotropic diffuse radiation, the perturbed upwelling and downwelling fluxes are, respectively:

\[ \Delta F^+(z) = C^+_8(x,z,z,1,0) \sum_{k=1}^{8} c_k^+ \psi_k(x) + C^-_8(x,z,z,1,0) \sum_{k=1}^{8} c_k^- q_k(x) - \sum_{k=1}^{8} c_k^0 q_k(z) \]  

(22a)

\[ \Delta F^-(x) = C^+_8(x,z,x,0,1) \sum_{k=1}^{8} c_k^+ \psi_k(x) + C^-_8(x,z,x,0,1) \sum_{k=1}^{8} c_k^- q_k(x) - \sum_{k=1}^{8} c_k^0 q_k(x) \]  

(22b)

where the coefficients are given in Table 3. The functions \( \psi_k \) and \( q_k \) are the same as that calculated for direct solar illumination. For these i.r. problems, we use both the exponential approximation as above and the linear approximation equivalent

\[ F^+(z) \approx F^+_o(z) - \Delta F^+(z) \]  

(22c)

\[ F^-(x) \approx F^-_o(x) - \Delta F^-(x) \]  

(22d)

The superior nature of this linear approximation for infrared transfer is discussed below.

<table>
<thead>
<tr>
<th>Thermal source</th>
<th>( c_1' = \delta r A_1 \delta t A_1 )</th>
<th>( c_1' = \delta r A_1 \delta t A_2 )</th>
<th>( c_1' = \delta t V_{11} \delta t V_{22} )</th>
<th>( c_1' = \delta t V_{12} \delta t V_{12} )</th>
<th>( c_5' = \delta t A_1 \delta t A_1 )</th>
<th>( c_6' = \delta t A_1 \delta t A_2 )</th>
<th>( c_7' = \delta t V_{21} \delta t V_{21} )</th>
<th>( c_8' = \delta t V_{11} \delta t V_{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 = \left( p_1 - \frac{m_1}{m_1 + c_1} \right) F_{lb} + \left( p_1 - \frac{m_1}{m_1 + c_1} \right) V_{11} + b_1 (Q^+_o + Q^-<em>o) / m</em>{1+3} )</td>
<td>( A_2 = \left( p_2 - \frac{m_1}{m_1 + c_1} \right) F_{lb} + \left( p_2 - \frac{m_1}{m_1 + c_1} \right) V_{11} - b_1 (Q^+_o + Q^-<em>o) / m</em>{1+3} )</td>
<td>( A_3 = \left( p_3 - \frac{m_1}{m_1 + c_1} \right) F_{lb} + \left( p_3 - \frac{m_1}{m_1 + c_1} \right) V_{11} + b_1 (Q^+_o + Q^-<em>o) / m</em>{1+3} )</td>
<td>( A_4 = \left( p_4 - \frac{m_1}{m_1 + c_1} \right) F_{lb} + \left( p_4 - \frac{m_1}{m_1 + c_1} \right) V_{11} + b_1 (Q^+_o + Q^-<em>o) / m</em>{1+3} )</td>
<td>( V_{11} = \left( -B_0 (Q^+_o + Q^-_o) + B_1 (Q^+_o + Q^-_o) / k^2 \right) )</td>
<td>( V_{12} = \left( -B_0 (Q^+_o + Q^-_o) - B_1 (Q^+_o + Q^-_o) / k^2 \right) )</td>
<td>( V_{21} = \left( -B_1 (Q^+_o + Q^-_o) / k^2 \right) )</td>
<td>( V_{22} = \left( -B_1 (Q^+_o + Q^-_o) / k^2 \right) )</td>
<td>( Q^+<em>o = \left( m</em>{1+3} - c_1 \right) )</td>
</tr>
</tbody>
</table>

Table 3. The coefficients required to perform adjoint perturbation calculations for the Planck source function. With the exception of coefficients \( c'(1) \ldots c'(8) \), all of the other coefficients are calculated once.
4. COMPARISONS WITH TWO-STREAM RESULTS

4.1. Results for direct illumination

The results of the perturbation calculations are presented here in terms of the absolute flux errors which are derived as the difference between fluxes computed using a standard two-stream Eddington model and the flux obtained using the perturbation method for a given base state. All fluxes are presented relative to an incident unit flux. The relative errors in the upwelling and downwelling fluxes were computed from $1 - \frac{F^\pm}{F^\pm_{\text{true}}}$ where $F^\pm_{\text{true}}$ are the true fluxes computed by the Eddington model.

Figures 3a and 3b present the absolute error in downwelling and upwelling fluxes respectively for an overhead sun ($\theta_0 = 0^\circ$) and for a base state single scattering albedo of 0.994 and optical depth of 10. The perturbed extinction and scattering cross sections are chosen so that the base state single scattering albedo is maintained through their range of variation. For downwelling fluxes, the errors are within 5% of the unit incident flux with the exception of the regions labelled "A" and "B". The absolute errors in the perturbed upwelling fluxes tend to be larger than for downwelling fluxes as indicated by the smaller region of 5% or less error. For optically thin media as would be found in region "C" in Fig. 3a and the dark shaded region in the lower...
Fig. 3. (a) Differences between downwelling fluxes calculated by the standard two-stream solution and the adjoint perturbation method are plotted as error contours. The solar zenith angle is $0^\circ$. Perturbations in the base state absorption and scattering were selected to constrain the single scattering albedo to 0.994. The underlying surface albedo was $A_s = 0$ and the asymmetry parameter set to $g = 0.85$. (b) Similar to (a) except that the error contours are for upwelling flux differences.

left corner of Fig. 3b, the Eddington approximation breaks down, yielding negative fluxes. In that case the perturbed fluxes will also be non-physical, i.e., the error is in the Eddington Approximation and not in the perturbation method.

To explore further how the perturbation approximation is affected by deviations from the base state, the absolute error in the downwelling and upwelling fluxes are given in Figs 4a and 4b when both the absorption and scattering are allowed to vary according to (17). For downwelling fluxes, the perturbation fluxes are accurate over most of the parameter space with regions A–E exhibiting errors of less than 6%. In region “A” of Fig. 4b, the absolute error in the perturbed upwelling flux ranges from nearly 2% to 11%. The perturbation $\delta_{\text{ sca}}$ is seen to vary from approximately $-70\%$ to $+50\%$ while the variation in $\delta_{\text{ abs}}$ extends from approximately $-100\%$ to $400\%$. For both upwelling and downwelling fluxes, the perturbation approach accommodates larger perturbations in absorption than in scattering (i.e., the contours tend to be vertically oriented in these diagrams). We propose an application to exploit this feature in the following section.
Since the base state for the solar radiation case also depends also on the solar zenith angle, the sensitivities of the perturbed method to this parameter requires investigation. The zenith angle chosen to illustrate the performance of the method is 60°. All base state parameters were as in the overhead sun case, and again with unit incident solar flux. Figures 5a and 5b confirm the performance of the method at this solar zenith angle with errors ranging from nearly -3% to under 7% in regions A–E for downwelling fluxes. The range of δ_{ext} over which the perturbations for upwelling fluxes are tolerable extend to almost 100% thus exceeding that for the normal incidence example as in Fig. 3b.

Departures of the perturbed downwelling fluxes from those calculated using the conventional two-stream approximation for \( \theta_0 = 60° \) are illustrated in Fig. 6a when both the extinction and scattering cross sections are permitted variation according to (17). It is observed that in the regions marked A to F, the errors in downwelling flux (Fig. 6a) are less than 7%, corresponding to tolerable perturbations in \( \sigma_{abs} \) exceeding 400%, and in \( \sigma_{sca} \), exceeding more than 300%. The absolute errors shown in Fig. 6b for reflected flux again show that larger perturbations in both \( \sigma_{sca} \) and \( \sigma_{abs} \) can be tolerated (region A) by comparison to region A of Fig. 4b.

4.2. Results for diffuse illumination

This section explores the accuracies of linear and exponential corrections to the base state fluxes for emitting media illuminated by diffuse radiation at the lower boundary. In the discus-
sion to follow, only the base state absorption and scattering were perturbed while the layer was taken to be isothermal. The isothermal constraint was borne from the desire to keep the comparisons simple and is strictly not necessary as the solutions allow for linear temperature profiles. The temperature of the medium was selected to be 248.5 K and was illuminated from below by a source emitting diffuse radiation at a temperature of 288 K. The asymmetry factor was fixed at $g = 0.85$ while the base state single scattering albedo and optical depth were $\omega = 0.60$ and $\tau = 5$ respectively. The value of the single scatter albedo is typical of cloud droplets at an infrared wavelength of 10 $\mu$m. At this wavelength the Planck function at 248.5 K has a value of 0.45 when normalized by the incident Planck function at 288 K.

Similar to the previous results, the results for the diffuse illumination problem are also presented in terms of flux differences obtained using the usual two-stream solution and the perturbation solutions. Two sets of flux differences are presented, one set of upwelling and
downwelling differences obtained for perturbations solutions derived using the linear relation (Figs 7a and 7b) and a second set of flux differences (Figs 8a and 8b) derived using the exponential relationship. It is evident from these figures that the exponential corrections to the perturbation are much less tolerant to large departures from the base state than is this same approximation applied to direct illumination problem. Furthermore, the accuracy of the fluxes is high only along a very narrow and approximately horizontal region of these diagrams suggesting a tolerance to large perturbations in scattering but smaller perturbations in absorption. This contrasts with the solar case where large perturbations in the absorption can be accommodated. The linear relation serves the perturbation method better but again is accurate over a narrow swath of increasing scattering. The relatively poor performance of the exponential correction is thought to arise from the linear terms present in the exact solution. In the case of direct illumination, these terms are absent; the solution is entirely of exponential form and this suggests that an exponential correction be sought. In the present case, the linear approximation is more accurate because of the presence of the linear terms in the exact solution of the fluxes. It may be useful to compute higher order terms in the series given by (1) to improve the accuracy of the linear approximation, provided that the result will not slow the computational process significantly.

![Fig. 5(a)](image)
Fig. 5. (a) Similar to Fig. 3a except that the downwelling fluxes are computed for a solar zenith angle of 60°. (b) Similar to Fig. 3b except that the upwelling fluxes are computed for a solar zenith angle of 60°.

We remark in passing that results similar to these were obtained for isothermal media of temperatures 230 and 270 K with otherwise identical boundary conditions and optical properties to those used to produce the results of Figs 7 and 8. Furthermore it was observed that for a source function that decreases linearly with altitude as in the non-isothermal case, both the exponential and linear corrections increase in accuracy, but only marginally so.

5. EXAMPLE APPLICATION: COMPUTATION OF BROAD BAND FLUXES

The perturbation approach to radiative transfer has been developed and its utility explored in a series of articles by Box and colleagues (e.g., Loughlin and Box\textsuperscript{10}). The focus of these studies is on surface fluxes and perturbations were defined by changing the composition of the atmosphere (through changes in the ozone or aerosol profiles from given base-state profiles). Those studies show the approach to be surprisingly accurate over a large range of perturbations.

More recently, Box et al\textsuperscript{3} applied the technique to calculate spectrally integrated u.v. fluxes at the surface and introduced perturbations as a way of dealing with spectrally varying ozone absorption. We explore here a variation on this idea motivated in part by the results shown in
Adjoint perturbation method

Fig. 4 for the problem of collimated incident radiation. The results show how the perturbation approach is relatively more accurate over a large variation of absorption given fixed scattering properties. This suggests that the perturbation approach may be profitably applied to the problem of particle scattering embedded in a molecular absorbing atmosphere where gas absorption may be treated as perturbations defined in terms of changes in the absorption coefficient. Normally these problems are posed such that the molecular absorption in a selected spectral band $\Delta \nu$ is superimposed on a (spectrally constant) scattering continuum. The band averaged line absorption is commonly approximated by the $k$-distribution, correlated $k$-distribution method (Goody and Yung\(^{11}\)) or as a sum of exponentials such that the band transmission is given as:

$$ T(\nu) = \frac{1}{\Delta \nu} \int e^{-k(\nu)} d\nu \approx \sum_{i=1}^{N} w_i \exp(-k_i \nu) $$

(23)

where the weights $w_i = f(k_i) dk$ are the fractions of the band occupied by absorption coefficients $k_i$ to $k_i + dk$. Solutions are then obtained by summing the $N$ weighted solutions obtained with the $N$ absorption coefficients (e.g., Stephens\(^{12}\)), namely

(a)

Solar, F\(^{-}\) Abs. Error, $\theta_0=60^0$

$\sigma_{\text{abs}} = 0.06$, $\sigma_{\text{sca}} = 9.94$

CONTOUR FROM .0120 TO .1044 BY .0132

Fig. 6(a). Caption overleaf.
Fig. 6. (a) Similar to Fig. 4a except that the downwelling fluxes were computed for a solar zenith angle of 60°. (b) Similar to Fig. 4a except that the upwelling fluxes were computed for a solar zenith angle of 60°.

\[ F(z) = \sum_{i=1}^{N} w_i F_i(z) \]  

where \( F(z) \) is the flux derived from a radiative transfer solution (in this study these are the two-stream fluxes) corresponding to \( k_i \).

In order to apply the technique discussed above, the concept here is to define the perturbations in terms of a change in the absorption coefficient \( k_i \) (relative to a chosen base-state value \( k_o = k_{j_{\text{max}}} \)). The perturbation coefficients then follow as

\[ \Delta t_i = \frac{1}{4} \rho (k_i - k_{j_{\text{max}}}) \]  

\[ \Delta r_i = -\frac{1}{4} \rho (k_i - k_{j_{\text{max}}}) \]

where \( \rho \) is the density of air which is the conversion factor from a mass absorption to a volume absorption coefficient. In this definition, the base state condition is nominally chosen to corre-
spond to the absorption coefficient associated with the maximum weight \( w_i = \delta_{\text{imax}} \). The band integrated fluxes are then obtained as a weighted sum of the perturbation fluxes,

\[
F(z) = w_{\text{imax}} F_{\text{imax}} + \sum_{i=1,i \neq \text{imax}}^{N} w_i F_i
\]

where the first term is the weighted base state solution and the summation term contains the fluxes \( F_i \), obtained using (21). An obvious advantage of this application is that the perturbed flux solutions corresponding to the largest absorption perturbations are weighted least in this summation.

We now illustrate this application using the heuristic example of a single layer of atmosphere illuminated from above by collimated radiation and overlying a dark surface (so that surface reflections may be neglected). The atmosphere is composed of absorbing gases and includes conservatively scattering particles of optical depth \( \tau_c \) which is referred to as the continuum optical depth since this is taken to be independent of spectral wavelength for this example. The spectral properties of the gases are taken from the k-distribution model of Fu and Liou\(^{13}\) which divides the solar spectrum between 0.2 and 4.0 \( \mu \text{m} \) into six bands. The gaseous absorption of each band is represented in the form of (23), and Table 4 is reproduced from Fu and Liou\(^{13}\) for reference. This table also includes the number of coefficients \( N \) used for each band and the gases included
Fig. 7. (a) Upwelling i.r. flux difference contours between forward two-stream solutions and those obtained by the adjoint perturbation method using a linear correction. The base state optical depth is 5 with a single scattering albedo of $\omega = 0.65$ at $\lambda = 10 \mu m$. The Planck function of the (isothermal) medium at 248.5 K has a value of 0.45 when normalized by the incident flux at 288 K. The asymmetry factor $g$ is 0.85. (b) As in (a) except that the flux difference contours are for i.r. flux differences.

Two sets of computations were carried out. The first applies the two-stream model to the atmosphere to calculate the broad band fluxes through the formal k-distribution approach. This requires 56 radiative transfer solutions to yield the spectrally integrated fluxes (this number is the sum of the number of ks in each band as given in Table 4). The second set of computations applies the perturbation approach described above where a base state two-stream solution is carried out once per band (hence requiring just six solutions) and the perturbations derived according to (19a)–(19b) are used to approximate the fluxes associated with the remaining ks.

The resulting broad band fluxes derived from these computations are differenced and plotted in Fig. 9 as a function of $\tau_c$. Flux differences are expressed in W m$^{-2}$ and are shown for reflected flux (solid), transmitted flux (dashed) and the absorbed flux (dotted). The incident flux is 670 W m$^{-2}$ which corresponds to a collimated source of 1340 W m$^{-2}$ inclined at a solar zenith angle of 60°. For reference, the clear sky reflected, transmitted and absorbed fluxes are, respectively, 48, 496 and 126 W m$^{-2}$. The flux differences are shown to be of the order of 5 W m$^{-2}$.
except for reflection and absorption from optically thick continua. The errors of the reflected flux grow as the optical depth of the scattering continuum increases but the transmitted flux remains accurate (in an absolute sense) throughout.

These results are presented here merely as an illustration of the potential utility of the perturbation approach to this class of problems. No attempt has been made to explore ways of reducing the reflection errors that occur under large optical depth conditions. It is worth mentioning that the characteristics of the errors shown can be altered slightly by fine tuning the selection of the base state.

6. SUMMARY AND CONCLUSIONS

This paper develops highly efficient computational techniques for solving for the two-stream fluxes based on the adjoint perturbation approximation. The theoretical foundations of these approximations was known to workers in the field of nuclear physics, and more recently has been applied to problems of atmospheric radiative transfer by Gerstl and Box. We extend this previous work by (a) developing new closed form analytical expressions for the perturbation two-stream fluxes for both solar and thermal source functions in a single plane parallel layer and (b) applying the perturbation method to calculate broad band fluxes emerging from a single atmospheric layer.

Two kinds of perturbation corrections to the base state fluxes were explored, one relevant to solar transfer problems and another relevant to thermal transfer problems. In the case of solar

\[
\delta_{\text{scs}} = \begin{bmatrix}
\alpha & 1 \\
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \\
1 & 0
\end{bmatrix}
\]

Fig. 8(a). Caption overleaf.
radiation, the exponential correction conjectured by Box\textsuperscript{4} was, in general, considerably more accurate than the linear correction when departures from the base state optical properties were large. The application of the exponential correction to broad band calculations in which the gas absorption coefficients of the k-distribution method were treated as the perturbations shows that errors in the fluxes can be comparable or smaller to those of the Eddington two-stream fluxes themselves. Furthermore, the exponential correction yields more accurate results in the transmission than in the albedo. The accuracy is dependent on the base state optical properties as well as on the solar zenith angle. With respect to the latter, greatest accuracy is obtained for angles greater than about 30 degrees regardless of the choice of the base state optical properties.

Table 4. Spectral division used in the calculation of broadband solar radiation

<table>
<thead>
<tr>
<th>Spectral region (cm\textsuperscript{-1})</th>
<th>No. of spectral lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>2500–2850</td>
<td>10</td>
</tr>
<tr>
<td>2850–4000</td>
<td>8</td>
</tr>
<tr>
<td>4000–5250</td>
<td>12</td>
</tr>
<tr>
<td>5250–7700</td>
<td>7</td>
</tr>
<tr>
<td>7700–14,500</td>
<td>12</td>
</tr>
<tr>
<td>&gt;14,500</td>
<td>5</td>
</tr>
<tr>
<td>Total</td>
<td>54</td>
</tr>
</tbody>
</table>
In the case of thermal radiative transfer, neither the exponential or linear corrections yielded accurate fluxes when the deviations from the base state optical properties were large. More specifically, large departures from the base state absorption (greater than 20%) resulted in large errors in the computed fluxes. If the perturbations in the absorption are small, however, then very large perturbations (e.g., 400%) in the scattering can be accommodated with little error incurred in the fluxes. Thus thermal radiative transfer differs from that in the solar since in the latter, large perturbations in both the scattering and absorption can be accommodated.

The development presented here serves as the foundation for further research to extend these results to multi-layered atmospheres and to provide improvements in extending the utility of perturbation methods to a broader range of infrared transfer problems.

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REFERENCES

APPENDIX A

Basic Two-stream Solutions

The fundamental solution of the two-stream equations can be expressed in terms of the matrix exponential

\[ M(z - y) = e^{-A(z - y)} \]  (A1a)

(refer to Fig. 1 for the definition of \( y \) and \( z \) which is sometimes referred to as the mapping or propagation function (e.g., Mobley\textsuperscript{15} and Flatau and Stephens\textsuperscript{16}) and has the property

\[ \left[e^{-A(z - y)}\right]^{-1} = e^{A(z - y)} \]

The block structure of \( A \) as in (5a) leads to a similar block structure in this mapping matrix, namely

\[ m(z - y) = \begin{pmatrix} m_{ss}(z - y) & m_{se}(z - y) \\ m_{es}(z - y) & m_{ee}(z - y) \end{pmatrix} \]

The two-stream version of matrix \( A \) leads to the following analytic form for the elements of this matrix, (Flatau and Stephens\textsuperscript{16})

\[ m_{ss}(z - y) = \frac{e^{k_2(z - y)}}{2} f_+ + \frac{e^{-k_2(z - y)}}{2} f_- \]
\[ m_{se}(z - y) = \frac{e^{k_2(z - y)}}{2} f_+ - \frac{e^{-k_2(z - y)}}{2} f_- \]
\[ m_{es}(z - y) = \frac{e^{-k_2(z - y)}}{2} f_+ + \frac{e^{k_2(z - y)}}{2} f_- \]
\[ m_{ee}(z - y) = \frac{e^{-k_2(z - y)}}{2} f_+ - \frac{e^{k_2(z - y)}}{2} f_- \]

where

\[ f_+ = (1 \pm t/k) \]

and \( k^2 = t^2 - r^2 \).

The general two-stream solutions follow from (4a) after multiplying both sides of this equation by the exponential \( e^{Ax} \) of the matrix leading to

\[ e^{-Ay} \frac{dF}{dy} - e^{-Ay} A F(y) = e^{-Ay} Q(y) \]  (A2)

which on integration from \( x \to z \) yields

\[ F(z) = e^{-A(z-x)} F(x) + S(z,x) \]  (A3)

where

\[ S(z,x) = \int_x^z e^{-A(z-y)} Q(y) dy \]  (A4)

Consider the classical radiative transfer problem that seeks the emergent fluxes leaving an isolated layer that lies between level \( z \) and level \( x \) in terms of both incident fluxes and sources within the layer. Equation (A3) resembles the traditional integral form of the radiative transfer equation solution. However, it contains the desired emergent fluxes (i.e., the solution) on both sides of the equation as seen more clearly in the expanded form

\[ \begin{pmatrix} F^+(x) \\ F^-(x) \end{pmatrix} = \begin{pmatrix} m_{ss}(z-x) & m_{se}(z-x) \\ m_{es}(z-x) & m_{ee}(z-x) \end{pmatrix} \begin{pmatrix} F^+(z) \\ F^-(z) \end{pmatrix} + \begin{pmatrix} S^+(z,x) \\ S^-(z,x) \end{pmatrix} \]  (A5)

Reorganization of (A5) in its interaction form gives the desired emergent fluxes, \( F^+(z) \) and \( F^-(z) \), in terms of input fluxes and sources. For example, consider a sourceless medium

\[ \begin{pmatrix} F^+(x) \\ F^-(-x) \end{pmatrix} = \begin{pmatrix} 1/m_{ss} & -m_{se}/m_{ss} \\ m_{es}/m_{ee} & m_{ee} - m_{es}m_{se}/m_{ee} \end{pmatrix} \begin{pmatrix} F^+(x) \\ F^-(x) \end{pmatrix} \]  (A6)

where the notation indicating that the mapping factors are defined for the layer \((z,x)\) is dropped for convenience. This
Adjoint perturbation method identifies the global layer diffuse reflection and transmission functions as

\[
R(z,x) = \frac{-m_{++}(z - x)}{m_{++}(z - x)} \quad (A7a)
\]

\[
T(z,x) = \frac{1}{m_{++}(z - x)} \quad (A7b)
\]

Consider the medium with sources illuminated with zero incident fluxes (known as vacuum boundary conditions). Then we obtain

\[
\begin{pmatrix}
F^+(z) \\
F^-(z)
\end{pmatrix} = \begin{pmatrix}
-S^+(z,x|m_{++}(z - x)) \\
-m_{--}(z - x)S^+(z,x|m_{++}(z - x)) + S^-(z,x)
\end{pmatrix}
\]

(A8)

which are formally the particular solutions to (A3) for general solutions. When we write (A8) as

\[
F^+(w = z) = \int_0^w \mathcal{F}^+(y,z)Q^+(y) + \mathcal{F}^-(y,z)Q^-(y) \, dy
\]

(A9a)

\[
F^-(w = x = 0) = \int_0^x \mathcal{F}^+(y,x)Q^+(y) + \mathcal{F}^-(y,x)Q^-(y) \, dy
\]

(A9b)

where the adjoint fluxes

\[
\mathcal{F}^+(y,z) = \frac{m_{++}(y)}{m_{++}(z)} - m_{--}(y - z)H(y - z)
\]

\[
\mathcal{F}^-(y,z) = \frac{m_{++}(y)}{m_{++}(z)} - m_{--}(y - z)H(y - z)
\]

\[
\mathcal{F}^+(y,x) = \frac{m_{++}(y)}{m_{++}(x)} m_{--}(y) - m_{--}(y - x)
\]

\[
\mathcal{F}^-(y,x) = \frac{m_{--}(y)}{m_{++}(x)} m_{--}(y) - m_{--}(y - x)
\]

become directly visible in these solutions.

**Solution With Solar And Infrared Sources**

We now specify a \( y \) dependence to our source function which will enable the analytic evaluation of the integrals of (A8). The first source to be considered is the solar source which we write in the form

\[
Q_{\odot}(y) = \left( \frac{X^+}{X^-} \right) e^{-\sigma_{\text{ext}}(z-y)/\mu}\theta\quad (A10)
\]

where the coefficients \( X^\pm \) are also defined in Table 2. Given the functional form of (A10) and the form of the mapping functions (A14), then the integral functions (A4) become

\[
S^+(z,y) = -\frac{1}{C_+} \left[ C_+ \left( X^+ \odot f_+ - \frac{r}{k} X^+ \odot f_- \right) + C_- \left( X^+ \odot f_+ + \frac{r}{k} X^- \odot f_- \right) \right]
\]

\[
S^-(z,y) = -\frac{1}{C_-} \left[ C_+ \left( X^+ \odot f_+ + X^- \odot f_- \right) + C_- \left( X^+ \odot f_+ - \frac{r}{k} X^- \odot f_- \right) \right]
\]

(A11)

where

\[
C_+ = \frac{1}{k - \sigma_{\text{ext}}/\mu} \left[ e^{\delta(z-y)} - e^{-\sigma_{\text{ext}}(z-y)/\mu} \right]
\]

\[
C_- = \frac{1}{k - \sigma_{\text{ext}}/\mu} \left[ e^{-\delta(z-y)} - e^{-\sigma_{\text{ext}}(z-y)/\mu} \right]
\]

are determined from the boundary conditions. The second source considered is the infrared source which we take to be a linear function of \( y \) as opposed to the usual practice in making this source a linear function of the optical depth \( \tau \). For this source,

\[
Q_\nu(y) = \sigma_{\text{ext}}[\delta(z) + \Delta \delta(z - y)]
\]

(A12)

where

\[
\Delta \delta = \frac{\delta(z) - \delta(y)}{z - y}
\]

Again combination of (A12) with (A8) and (A4) yields

\[
S^+(z,y) = \frac{\sigma_{\text{abs}}}{2k} \left[ C_+ \left( \delta(y) - \Delta \delta \right) e^{\mu C} - C_+ \left( \delta(1) + \Delta \delta \right) e^{-\mu C} + C_+ \left( -\frac{\Delta \delta}{k} \right) + C_+ \left( \delta(z) + \frac{\Delta \delta}{k} \right) \right]
\]

(A13a)
\[ S^{-}(x,y) = \frac{\sigma_{\text{abs}}}{2k} \left[ C_1 \left( \mathcal{A}(y) - \frac{\Delta \mathcal{A}}{\kappa} \right) e^{x} - C_1 \left( \mathcal{A}(y) + \frac{\Delta \mathcal{A}}{\kappa} \right) e^{-x} + C_1 \left( -\mathcal{A}(x) + \frac{\Delta \mathcal{A}}{\kappa} \right) + C_1 \left( \mathcal{A}(x) + \frac{\Delta \mathcal{A}}{\kappa} \right) \right], \quad (A13b) \]

where

\[ C_1 = f_+ - \frac{r}{\kappa}, \]
\[ C_2 = f_- + \frac{r}{\kappa}. \]